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Analytic Spin and Pseudospin Solutions to the Dirac Equation for the Quadratic Exponential-Type Potential Plus Eckart Potential and Yukawa-Like Tensor Interaction

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Abstract: We solve the Dirac equation for the quadratic exponential-type potential plus Eckart potential including a Yukawa-like tensor potential with arbitrary spin–orbit coupling quantum number κ. In the framework of the spin and pseudospin (pspin) symmetry, we obtain the energy eigenvalue equation and the corresponding eigenfunctions in closed form by using the Nikiforov–Uvarov method. Also Special cases of the potential as been considered and their energy eigen values as well as their corresponding eigen functions are obtained for both relativistic and non-relativistic scope.

Keywords: Dirac Equation, Quadratic Exponential-Type Potential, Eckart Potential, Spin and Pseudospin Symmetry, Nikiforov-Uvarov Method

1. Introduction

The Dirac and Klein–Gordon wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics. In recent years, a lot of effort has been put into solving these relativistic wave equations for various potentials by using different methods. For example, some authors assumed that the scalar potential is equal to the vector potential and obtained the exact solutions of the Dirac equation with some typical potential by using different methods. These investigations include the harmonic oscillator [1], Eckart potential [2, 3], Woods–Saxon potential [4], Hulthen potential [5], ring-shaped Kratzer-type potential [6], double ring-shaped oscillator [7], Hartmann potential [8, 9], Rosen–Morse-type potential [10], generalized symmetrical double-well potential [11], Scarf-type potential [12] and ring-shaped non-spherical oscillator [13]. Our research groups have recently investigated the eigensolutions (eigenvalues and eigenfunctions) of Klein-Gordon, Dirac and Schrodinger equations using superposed or mixed potentials. Some notable ones includes Woods-Saxon plus Attractive Inversely Quadratic potential (WSAIQP) [14], Manning-Rosen plus a class of Yukawa potential (MRCYP) [15], generalized wood-Saxon plus Mie-type potential (GWSMP) [16], Kratzer plus Reduced Pseudoharmonic Oscillator potential (KRPHOP) [17], Inversely Quadratic Yukawa plus attractive radial potentials (IQYARP) [18], Modified Echart plus Inverse Square Molecular Potentials (MEISP) [19].

These methods include the supersymmetry quantum mechanics [20, 21], the Nikiforov–Uvarov (NU) method [22], the asymptotic iteration method [23] and others. Recently, there has been renewed interest in solving simple quantum mechanical systems within the framework of the NU method. This algebraic technique is based on solving the second-order linear differential equations, which has been used successfully to solve the Schrödinger, Dirac and Klein–Gordon wave equations in the presence of some well-known potentials [24].

In this work, our aim is to solve the Dirac equation for

Quadratic exponential-type potential plus Eckart potential (QEPE) potential in the presence of spin and pspin symmetries and by including a Yukawa-like tensor potential. The QEPE potential takes the following form:

$$V(r) = D\left[\frac{ae^{2\alpha r}+be^{\alpha r}+c}{(e^{\alpha r}-1)^2}\right] - A\frac{e^{-\alpha r}}{(1-e^{-\alpha r})} + B\frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2} \quad (1)$$

Thus eq. (1a) can be further expressed as

$$V(r) = D\left[\frac{a+be^{-\alpha r}+ce^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] - A\frac{e^{-\alpha r}}{(1-e^{-\alpha r})} + B\frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2} \quad (2)$$

where α is the range of the potential, D, A, B are potential depths and a, b, c are adjustable parameters. This potential is known as an analytical potential model and is used for the vibrational energy of diatomic molecules.

This paper is organized as follows. In section 2, we briefly introduce the Dirac equation with scalar and vector potentials with arbitrary spin–orbit coupling quantum number κ including tensor interaction under spin and pspin symmetry limits. The Nikiforov–Uvarov (NU) method is presented in section 3. The energy eigenvalue equations and corresponding eigenfunctions are obtained in section 4. In this section, In section 5, we discussed some special cases of the potential. Finally, our conclusion is given in section 6.

2. The Dirac Equation with Tensor Coupling Potential

The Dirac equation for fermionic massive spin-1/2 particles moving in the field of an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$ and a tensor potential $U(r)$ (in units ħ = c = 1) is

$$[\vec{\alpha}\cdot\vec{p} + \beta(M + S(r)) - i\beta\vec{\alpha}\cdot\vec{r}U(r)]\psi(\vec{r}) = [E - V(r)]\psi(\vec{r}). \quad (3)$$

where E is the relativistic binding energy of the system, $p = -i\vec{\nabla}$ is the three-dimensional momentum operator and M is the mass of the fermionic particle. $\vec{\alpha}$ and β are the 4×4 usual Dirac matrices given by

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (4)$$

where I is the 2×2 unitary matrix and $\vec{\sigma}$ are three-vector spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (5)$$

The eigenvalues of the spin–orbit coupling operator are $\kappa = \left(j+\frac{1}{2}\right) > 0$ and $\kappa = -\left(j+\frac{1}{2}\right) < 0$ for unaligned spin $j = l - \frac{1}{2}$ and aligned spin $j = l + \frac{1}{2}$, respectively. The set (H^2, K, J^2, J_z) can be taken as the complete set of conservative quantities with $\vec{J}$ being the total angular momentum operator and $K = (\vec{\sigma}.\vec{L} + 1)$ is the spin–orbit where $\vec{L}$ is the orbital angular momentum of the spherical nucleons that commutes with the Dirac Hamiltonian. Thus, the spinor wave functions can be classified according to their angular momentum j, the spin–orbit quantum number κ and the radial quantum number n. Hence, they can be written as follows:

$$\psi_{n,\kappa}(\vec{r}) = \begin{pmatrix} f_{n,\kappa}(\vec{r}) \\ g_{n,\kappa}(\vec{r}) \end{pmatrix} = \frac{1}{r}\begin{pmatrix} F_{n,\kappa}(r) & Y_{jm}^{l}(\theta,\varphi) \\ iG_{n,\kappa}(r) & Y_{jm}^{\tilde{l}}(\theta,\varphi) \end{pmatrix}, \quad (6)$$

where $f_{n,\kappa}(\vec{r})$ is the upper (large) component and $g_{n,\kappa}(\vec{r})$ is the lower (small) component of the Dirac spinors. $Y_{jm}^{l}(\theta,\varphi)$ and $Y_{jm}^{\tilde{l}}(\theta,\varphi)$ are spin and pspin spherical harmonics, respectively, and m is the projection of the angular momentum on the $z-axis$. Substituting equation (6) into equation (3) and making use of the following relations

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B}), \quad (7)$$

$$(\vec{\sigma}\cdot\vec{P}) = \vec{\sigma}\cdot\hat{r}\left(\hat{r}\cdot\vec{P} + i\frac{\vec{\sigma}\cdot\vec{L}}{r}\right), \quad (8)$$

together with the properties

$$(\vec{\sigma}\cdot\vec{L})Y_{jm}^{\tilde{l}}(\theta,\varphi) = (\kappa-1)Y_{jm}^{\tilde{l}}(\theta,\varphi),$$
$$(\vec{\sigma}\cdot\vec{L})Y_{jm}^{l}(\theta,\varphi) = -(\kappa-1)Y_{jm}^{l}(\theta,\varphi), \quad (9)$$
$$(\vec{\sigma}\cdot\hat{r})Y_{jm}^{\tilde{l}}(\theta,\varphi) = -Y_{jm}^{l}(\theta,\varphi),$$
$$(\vec{\sigma}\cdot\hat{r})Y_{jm}^{l}(\theta,\varphi) = -Y_{jm}^{\tilde{l}}(\theta,\varphi),$$

one obtains two coupled differential equations whose solutions are the upper and lower radial wave functions $F_{n,\kappa}(r)$ and $G_{n,\kappa}(r)$ as

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n,\kappa}(r) = (M + E_{n\kappa} - \Delta(r))\,G_{n,\kappa}(r), \quad (10)$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G_{n,\kappa}(r) = (M - E_{n\kappa} + \Sigma(r))\,F_{n,\kappa}(r), \quad (11)$$

where

$$\Delta(r) = V(r) - S(r), \quad (12)$$

$$\Sigma(r) = V(r) + S(r), \quad (13)$$

After eliminating $F_{n,\kappa}(r)$ and $G_{n,\kappa}(r)$ in equations (8), we obtain the following two Schrodinger-like differential equations for the upper and lower radial spinor components:

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - \frac{dU(r)}{dr} - U^2(r)\right]F_{n,\kappa}(r) + \frac{\frac{d\Delta(r)}{dr}}{M+E_{n\kappa}-\Delta(r)}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n,\kappa}(r) = [(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))]F_{n,\kappa}(r) \quad (14)$$

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) + \frac{dU(r)}{dr} - U^2(r)\right]G_{n,\kappa}(r) + \frac{\frac{d\Sigma(r)}{dr}}{M-E_{n\kappa}+\Sigma(r)}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G_{n,\kappa}(r) = [(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))]\,G_{n,\kappa}(r), \quad (15)$$

respectively, where $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$ and $\kappa(\kappa+1) = l(l+1)$.

The quantum number κ is related to the quantum numbers for spin symmetry l and pspin symmetry $\hat{l}$ as

$$\kappa = \begin{cases} -(l+1) = -\left(j+\frac{1}{2}\right)\left(s_{1/2}, p_{3/2}, etc\right) \\ j = l+\frac{1}{2}, aligned\ spin\ (\kappa < 0), \\ +l = +\left(j+\frac{1}{2}\right)\left(p_{1/2}, d_{3/2}, etc\right) \\ j = l-\frac{1}{2}, unaligned\ spin\ (\kappa > 0), \end{cases} \quad (16)$$

and the quasidegenerate doublet structure can be expressed in terms of a pspin angular momentum $\hat{s} = 1/2$ and pseudo-orbital angular momentum $\hat{l}$, which is defined as

$$\kappa = \begin{cases} -\hat{l} = -\left(j+\frac{1}{2}\right)\left(s_{1/2}, p_{3/2}, etc\right) \\ j = \hat{l}-\frac{1}{2}, aligned\ spin\ (\kappa < 0), \\ +(\hat{l}+1) = +\left(j+\frac{1}{2}\right)\left(d_{3/2}, f_{5/2}, etc\right) \\ j = \hat{l}+\frac{1}{2}, unaligned\ spin\ (\kappa > 0), \end{cases} \quad (17)$$

where $\kappa = \pm 1, \pm 2, \ldots$. For example, $(1s_{1/2}, 0d_{3/2})$ and $(0p_{3/2}, 0f_{5/2})$ can be considered as spin doublets

2.1. Spin Symmetry Limit

In the spin symmetry limit, $\frac{d\Delta(r)}{dr} = 0$ or $\Delta(r) = C_s$ =constant, with $\Sigma(r)$ taking as the QEPE potential eq. (2) and the Yukawa-like tensor potential. i.e

$$\Sigma(r) = V(r) = D\left[\frac{a+be^{-\alpha r}+ce^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] - A\frac{e^{-\alpha r}}{(1-e^{-\alpha r})} + B\frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2}, \quad (18)$$

$$U(r) = -\frac{H}{r}e^{-\alpha r}, \quad (19)$$

Under this symmetry, equation (14) is recast in the simple form

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - \frac{2\kappa H}{r^2} - \frac{H}{r^2} - \frac{H^2}{r^2}\right]F_{n,\kappa}(r) = \left[\gamma\left(D\left[\frac{a+be^{-\alpha r}+ce^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] - A\frac{e^{-\alpha r}}{(1-e^{-\alpha r})} + B\frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2}\right) + \beta^2\right]F_{n,\kappa}(r) \quad (20)$$

where $\kappa = l$ and $\kappa = -l-1$ for κ < 0 and κ > 0, respectively. Also, $\gamma = (M + E_{n\kappa} - C_s)$ and $\beta^2 = (M - E_{n\kappa})(M + E_{n\kappa} - C_s)$. (21)

2.2. Pseudospin Symmetry Limit

Ginocchio[] showed that there is a connection between pspin symmetry and near equalityof the time component of a vector potential and the scalar potential, $V(r) \approx -S(r)$. After that, Meng et al [,] derived that if $\frac{d\Sigma(r)}{dr} = 0$ or $\Sigma(r) = C_{ps}$ =constant, then pspin symmetry is exact in the Dirac equation. Here, we are taking $\Delta(r)$ as the QEPE potential eq. (1) and the tensor potential as the Yukawa-like potential. thus, equation (11) is recast in the simple form

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{2\kappa H}{r^2} + \frac{H}{r^2} - \frac{H^2}{r^2}\right]G_{n,\kappa}(r) = \left[\tilde{\gamma}\left(D\left[\frac{a+be^{-\alpha r}+ce^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] - A\frac{e^{-\alpha r}}{(1-e^{-\alpha r})} + B\frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2}\right) + \tilde{\beta}^2\right]G_{n,\kappa}(r) \quad (22)$$

where $\kappa = -\tilde{l}$ and $\kappa = \tilde{l}+1$ for κ < 0 and κ > 0, respectively. Also, $\tilde{\gamma} = \left(E_{n\kappa} - M - C_{ps}\right)$ and $\tilde{\beta}^2 = (M + E_{n\kappa})\left(M - E_{n\kappa} + C_{ps}\right)$. (23)

to obtain the analytic solution, we use an approximation for the centrifugal term as []

$$\frac{1}{r^2} = \frac{\alpha^2}{(1-e^{-\alpha r})^2} \quad (24)$$

Finally, for the solutions to equations (21) and (22) with the above approximation, we will employ the NU method, which is briefly introduced in the following section

3. The Nikiforov–Uvarov Method

The NU method is based on the solutions of a generalized second order linear differential equation with special orthogonal functions. The hypergeometric NU method has shown its power in calculating the exact energy levels of all bound states for some solvable quantum systems.

$$\Psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\Psi_n'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)}\Psi_n(s) = 0 \quad (25)$$

Where σ(s) and $\bar{\sigma}$(s) are polynomials at most second degree and $\tilde{\tau}$(s) is first degree polynomials. The parametric generalization of the N-U method is given by the generalized hypergeometric-type equation

$$\Psi''(s) + \frac{c_1 - c_2 s}{s(1-c_3 s)}\Psi'(s) + \frac{1}{s^2(1-c_3 s)^2}[-\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3]\Psi(s) = 0 \quad (26)$$

Thus eqn. (3) can be solved by comparing it with equation (4) and the following polynomials are obtained

$$\tilde{\tau}(s) = (c_1 - c_2 s), \sigma(s) = s(1-c_3 s), \bar{\sigma}(s) = -\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3 \quad (27)$$

The parameters obtainable from equation (5) serve as important tools to finding the energy eigenvalue and eigenfunctions. They satisfy the following sets of equation respectively

$$c_2 n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n-1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \quad (28)$$

$$(c_2 - c_3)n + c_3n^2 - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_{.8}}) + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \quad (29)$$

While the wave function is given as

$$\Psi_n(s) = N_{n,l}S^{c_{12}}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}}P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3 s) \quad (30)$$

Where

$$c_4 = \frac{1}{2}(1-c_1), c_5 = \frac{1}{2}(c_2 - 2c_3), c_6 = c_5{}^2 + \epsilon_1, c_7 = 2c_4c_5 - \epsilon_2, c_8 = c_4{}^2 + \epsilon_3,$$

$$c_9 = c_3c_7 + c_3{}^2c_8 + c_6, c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c3\sqrt{c_8})$$

$$c_{12} = c_4 + \sqrt{c_8}, c_{13} = c_5 - (\sqrt{c_9} + c3\sqrt{c_8}) \quad (31)$$

and P_n is the orthogonal polynomials.

4. Solutions to the Dirac Equation

We will now solve the Dirac equation with the QEPE potential and tensor potential by using the NU method.

4.1. The Spin Symmetric Case

To obtain the solution to equation (20), by using the transformation $s = e^{-\alpha r}$, we rewrite it as follows:

$$\frac{d^2F_{n,\kappa}(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dF_{n,\kappa}(s)}{ds} + \frac{1}{s^2(1-s)^2}\Big[-\kappa(\kappa+1) - 2\kappa Hs - 2Hs + Hs^2 - H^2s^2 - \frac{\gamma}{\alpha^2}(Da + Dbs + Dcs^2 - As(1-s) + Bs) - \frac{\beta^2}{\alpha^2}(1-s)^2\Big]F_{n,\kappa}(s) = 0, \quad (32)$$

Eq. (32) is further simplified as

$$\frac{d^2F_{n,\kappa}(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dF_{n,\kappa}(s)}{ds} + \frac{1}{s^2(1-s)^2}\left[-(\frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Dc + \frac{\gamma}{\alpha^2}A + H^2 - H)s^2 + \left(\frac{2\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2}B + \frac{\gamma}{\alpha^2}A - \frac{\gamma}{\alpha^2}Db - 2\kappa H - 2H\right)s - \left(\frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Da + \kappa(\kappa+1)\right)\right]F_{n,\kappa}(s) = 0, \quad (33)$$

Comparing eq. (33) with eq. (26), we obtain

$$c_1 = 1, \epsilon_1 = \frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Dc + \frac{\gamma}{\alpha^2}A + H^2 - H$$

$$c_2 = 1, \epsilon_2 = \frac{2\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2}B + \frac{\gamma}{\alpha^2}A - \frac{\gamma}{\alpha^2}Db - 2\kappa H - 2H \tag{34}$$

$$c_3 = 1, \epsilon_3 = \frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Da + \kappa(\kappa+1)$$

and from eq. (31), we further obtain

$$c_4 = 0, c_5 = -\frac{1}{2},$$

$$c_6 = \frac{1}{4} + \frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Dc + \frac{\gamma}{\alpha^2}A + H^2 - H, c_7 = -\left(\frac{2\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2}B + \frac{\gamma}{\alpha^2}A - \frac{\gamma}{\alpha^2}Db - 2\kappa H - 2H\right),$$

$$c_8 = \frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Da + \kappa(\kappa+1), c_9 = \left(\eta_\kappa - \frac{1}{2}\right)^2 + \frac{\gamma}{\alpha^2}D(a+b+c) + \frac{\gamma}{\alpha^2}B, \text{ where } \eta_\kappa = \kappa + H + 1,$$

$$c_{10} = 1 + 2\sqrt{\frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Da + \kappa(\kappa+1)},$$

$$c_{11} = 2 + 2\left(\sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 + \frac{\gamma}{\alpha^2}D(a+b+c) + \frac{\gamma}{\alpha^2}B} + \sqrt{\frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Da + \kappa(\kappa+1)}\right), \tag{35}$$

$$c_{12} = \sqrt{\frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Da + \kappa(\kappa+1)},$$

$$c_{13} = -\frac{1}{2} - \left(\sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 + \frac{\gamma}{\alpha^2}D(a+b+c) + \frac{\gamma}{\alpha^2}B} + \sqrt{\frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Da + \kappa(\kappa+1)}\right)$$

In addition, the energy eigenvalue equation can be obtained by using eq. (29) as follows:

$$\left(n + \frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 + \frac{\gamma}{\alpha^2}D(a+b+c) + \frac{\gamma}{\alpha^2}B} + \sqrt{\frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Da + \kappa(\kappa+1)}\right)^2 = \frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Dc + \frac{\gamma}{\alpha^2}A + H^2 - H \tag{36}$$

By substituting the explicit forms of γ and β^2 after equation (20) into equation (36), one can readily obtain the closed form for the energy formula.

$$\left(n + \frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 + \frac{D(a+b+c)}{\alpha^2}(M + E_{n\kappa} - C_s) + \frac{B}{\alpha^2}(M + E_{n\kappa} - C_s)} + \sqrt{\frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa} - C_s)\right) + \frac{Da}{\alpha^2}(M + E_{n\kappa} - C_s) + \kappa(\kappa+1)}\right)^2 = \frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa} - C_s)\right) + \frac{Dc}{\alpha^2}(M + E_{n\kappa} - C_s) + \frac{A}{\alpha^2}(M + E_{n\kappa} - C_s) + H^2 - H \tag{37}$$

On the other hand, to find the corresponding wave functions, referring to equation (35) and eq. (30), we obtain the upper component of the Dirac spinor from eq. 24 as

$$F_{n,\kappa}(s) = B_{n,\kappa} s^{\sqrt{\frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Da + \kappa(\kappa+1)}} (1-s)^{\frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 + \frac{\gamma}{\alpha^2}D(a+b+c) + \frac{\gamma}{\alpha^2}B}} P_n^{\left(2\sqrt{\frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha^2}Da + \kappa(\kappa+1)}, 2\sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 + \frac{\gamma}{\alpha^2}D(a+b+c) + \frac{\gamma}{\alpha^2}B}\right)}(1-2s) \tag{38}$$

where $B_{n,\kappa}$ is the normalization constant. The lower component of the Dirac spinor can be calculated from equation (10)

$$G_{n,\kappa}(r) = \frac{1}{(M + E_{n\kappa} - C_s)}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n,\kappa}(r) \tag{39}$$

where $E_{n\kappa} \neq -M + C_s$.

4.2. The Pseudospin Symmetric Case

To avoid repetition in the solution of equation (22), we follow the same procedures explained in section 4.1 and hence obtain the following energy eigenvalue equation:

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{\tilde{\gamma}}{\alpha^2}D(a+b+c)+\frac{\tilde{\gamma}}{\alpha^2}B}+\sqrt{\frac{\tilde{\beta}^2}{\alpha^2}+\frac{\tilde{\gamma}}{\alpha^2}Da+\kappa(\kappa-1)}\right)^2=\frac{\tilde{\beta}^2}{\alpha^2}+\frac{\tilde{\gamma}}{\alpha^2}Dc+\frac{\tilde{\gamma}}{\alpha^2}A+H^2+H \quad (40)$$

By substituting the explicit forms of $\tilde{\gamma}$ *and* $\tilde{\beta}^2$after equation (23) into equation (40), one can readily obtain the closed form for the energy formula as

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{D(a+b+c)}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+\frac{B}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)}+\sqrt{\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa}+C_{ps})\right)+\frac{Da}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+\kappa(\kappa-1)}\right)^2=\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa}+C_{ps})\right)+\frac{Dc}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+\frac{A}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+H^2+H \quad (41)$$

and the corresponding wave functions for the upper Dirac spinor as

$$G_{n,\kappa}(r)=\tilde{B}_{n,\kappa}s^{\sqrt{\frac{\tilde{\beta}^2}{\alpha^2}+\frac{\tilde{\gamma}}{\alpha^2}Da+\kappa(\kappa-1)}}(1-s)^{\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{\tilde{\gamma}}{\alpha^2}D(a+b+c)+\frac{\tilde{\gamma}}{\alpha^2}B}}P_n^{\left(2\sqrt{\frac{\tilde{\beta}^2}{\alpha^2}+\frac{\tilde{\gamma}}{\alpha^2}Da+\kappa(\kappa-1)},2\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{\tilde{\gamma}}{\alpha^2}D(a+b+c)+\frac{\tilde{\gamma}}{\alpha^2}B}\right)}(1-2s) \quad (42)$$

were $\Lambda_\kappa=\ \kappa+H$ and $\tilde{B}_{n,\kappa}$ is the normalization constant. Finally, the Upper-spinor component of the Dirac equation can be obtained via equation (11) as

$$F_{n,\kappa}(r)=\frac{1}{(M-E_{n\kappa}+C_{ps})}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G_{n,\kappa}(r) \quad (43)$$

where $E_{n\kappa}\neq M+C_{ps}$.

5. Discussions

In this section, we are going to study some special cases of the energy eigenvalues given by Eqs. (37) and (41) for the spin and pseudospin symmetries, respectively.

Case 1. If one sets $C_s=0, C_{ps}=0, A=B=0$ in eq. (37) and eq. (41), we obtain the energy equation of quadratic exponential-type potential for spin and pseudospin symmetric Dirac theory respectively,

$$\left(n+\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2+\frac{D(a+b+c)}{\alpha^2}(M+E_{n\kappa})}+\sqrt{\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\frac{Da}{\alpha^2}(M+E_{n\kappa})+\kappa(\kappa+1)}\right)^2=\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\frac{Dc}{\alpha^2}(M+E_{n\kappa})+H^2-H \quad (44)$$

and

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{D(a+b+c)}{\alpha^2}(E_{n\kappa}-M)}+\sqrt{\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\frac{Da}{\alpha^2}(E_{n\kappa}-M)+\kappa(\kappa-1)}\right)^2=\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\frac{Dc}{\alpha^2}(E_{n\kappa}-M)+H^2+H \quad (45)$$

Case 2: If one sets $C_s=0, C_{ps}=0, D=0$ in eq. (37) and eq. (41), we obtain the energy equation of Eckart potential for spin and pseudospin symmetric Dirac theory respectively,

$$\left(n+\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2+\frac{B}{\alpha^2}(M+E_{n\kappa})}+\sqrt{\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\kappa(\kappa+1)}\right)^2=\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\frac{A}{\alpha^2}(M+E_{n\kappa})+H^2-H \quad (46)$$

and

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{B}{\alpha^2}(E_{n\kappa}-M)}+\sqrt{\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\kappa(\kappa-1)}\right)^2=\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\frac{A}{\alpha^2}(E_{n\kappa}-M)+H^2+H \quad (47)$$

Case 3: If one sets $C_s = 0, C_{ps} = 0, B = 0, D = 0$, in eq. (37) and eq. (41), we obtain the energy equation of Hulthen potential for spin and pseudospin symmetric Dirac theory respectively,

$$\left(n + \eta_\kappa + \sqrt{\frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa})\right) + \kappa(\kappa + 1)}\right)^2 = \frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa})\right) + \frac{A}{\alpha^2}(M + E_{n\kappa}) + H^2 - H \quad (48)$$

and

$$\left(n + \Lambda_\kappa + \sqrt{\frac{1}{\alpha^2}\left((M + E_{n\kappa})(M - E_{n\kappa})\right) + \kappa(\kappa - 1)}\right)^2 = \frac{1}{\alpha^2}\left((M + E_{n\kappa})(M - E_{n\kappa})\right) + \frac{A}{\alpha^2}(E_{n\kappa} - M) + H^2 + H \quad (49)$$

Case 4: If $A = B = 0, a = 1, b = -2(1+\delta), c = (1+\delta)^2$ and $\delta = e^{\propto r_e} - 1$, Eq. (1b) reduces to the generalized Morse potential

$$V(r) = D\left[\frac{1 - 2(1+\delta)e^{-\propto r} + e^{-2\propto r}}{(1 - e^{-\propto r})^2}\right] \quad (50)$$

from eq. (37) and eq. (41), if $C_s = 0, C_{ps} = 0$, we obtain the energy equation generalized Morse potential for spin and pseudospin symmetric Dirac theory respectively

$$\left(n + \frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 + \frac{D\delta^2}{\alpha^2}(M + E_{n\kappa})} + \sqrt{\frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa})\right) + \frac{D}{\alpha^2}(M + E_{n\kappa}) + \kappa(\kappa + 1)}\right)^2 = \frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa})\right) + \frac{D(1+\delta)^2}{\alpha^2}(M + E_{n\kappa}) + H^2 + H \quad (51)$$

and

$$\left(n + \frac{1}{2} + \sqrt{\left(\Lambda_\kappa - \frac{1}{2}\right)^2 + \frac{D\delta^2}{\alpha^2}(E_{n\kappa} - M)} + \sqrt{\frac{1}{\alpha^2}\left((M + E_{n\kappa})(M - E_{n\kappa})\right) + \frac{D}{\alpha^2}(E_{n\kappa} - M) + \kappa(\kappa - 1)}\right)^2 = \frac{1}{\alpha^2}\left((M + E_{n\kappa})(M - E_{n\kappa})\right) + \frac{D(1+\delta)^2}{\alpha^2}(E_{n\kappa} - M) + H^2 - H \quad (52)$$

Case 5: Let us now discuss the relativistic limit of the energy eigenvalues and wavefunctions of our solutions. If we take $C_s = 0, H = 0, \kappa \to l$ and put $S(r) = V(r) = \Sigma(r)$, the nonrelativistic limit of energy equation (37) and wave function (38) under the following appropriate transformations $M + E_{n\kappa} \to \frac{2\mu}{\hbar^2}$, and $M - E_{n\kappa} \to -E_{nl}$ becomes

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1) + \frac{2\mu D}{\alpha^2\hbar^2}(2a+b) - \frac{2\mu A}{\alpha^2\hbar^2} + \frac{2\mu B}{\alpha^2\hbar^2} + \left(n^2 + n + \frac{1}{2}\right) + (2n+1)\sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2\mu D}{\alpha^2\hbar^2}(a+b+c) + \frac{2\mu B}{\alpha^2\hbar^2}}}{(2n+1) + 2\sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2\mu D}{\alpha^2\hbar^2}(a+b+c) + \frac{2\mu B}{\alpha^2\hbar^2}}}\right]^2 - \frac{2\mu Da}{\alpha^2\hbar^2} - l(l+1)\right\} \quad (53)$$

and the associated wave functions $F_{n\kappa}(s) \to R_{n,l}(s)$ are

$$R_{n,l}(s) = N_{n,l}s^{U/2}(1-s)^{(V-1)/2}P_n^{(U,V)}(1-2s), \quad (54)$$

$$\text{where } U = 2\sqrt{\frac{2\mu E_{nl}}{\propto^2\hbar^2} + \frac{2\mu Da}{\alpha^2\hbar^2} + l(l+1)} \text{ and } V = 2\sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2\mu D}{\alpha^2\hbar^2}(a+b+c) + \frac{2\mu B}{\alpha^2\hbar^2}} \quad (55)$$

Case 6: If one $A = B = 0$ in eq. (53), we obtain the energy equation of quadratic exponential-type potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1) + \frac{2\mu D}{\alpha^2\hbar^2}(2a+b) + \left(n^2 + n + \frac{1}{2}\right) + (2n+1)\sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2\mu D}{\alpha^2\hbar^2}(a+b+c)}}{(2n+1) + 2\sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2\mu D}{\alpha^2\hbar^2}(a+b+c)}}\right]^2 - \frac{2\mu Da}{\alpha^2\hbar^2} - l(l+1)\right\} \quad (56)$$

Case 7: If $D = 0$ in eq. (53), we obtain the energy equation of the Eckart potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1) - \frac{2\mu A}{\alpha^2\hbar^2} + \frac{2\mu B}{\alpha^2\hbar^2} + \left(n^2 + n + \frac{1}{2}\right) + (2n+1)\sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2\mu B}{\alpha^2\hbar^2}}}{(2n+1) + 2\sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2\mu B}{\alpha^2\hbar^2}}}\right]^2 - l(l+1)\right\} \quad (57)$$

Case 8: If $B = 0, D = 0$ in eq. (53), we obtain the energy equation of the Hulthen potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1)-\frac{2\mu A}{\alpha^2\hbar^2}+\left(n^2+n+\frac{1}{2}\right)+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2}}\right]^2 - l(l+1)\right\} \quad (58)$$

Case 15: If $A = B = 0, a = 1, b = -2(1+\delta), c = (1+\delta)^2$ and $\delta = e^{\propto r_e} - 1$, Eq. (2) reduces to the generalized Morse potential

$$V(r) = D\left[\frac{1-2(1+\delta)e^{-\propto r}+e^{-2\propto r}}{(1-e^{-\propto r})^2}\right] \quad (59)$$

from eq. (53), we obtain the energy equation of generalized Morse potential

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1)-\frac{2\mu D\delta}{\alpha^2\hbar^2}+\left(n^2+n+\frac{1}{2}\right)+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu D\delta^2}{\alpha^2\hbar^2}}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu D\delta^2}{\alpha^2\hbar^2}}}\right]^2 - \frac{2\mu D}{\alpha^2\hbar^2} - l(l+1)\right\} \quad (60)$$

6. Conclusion

In the present paper, we solved the Analytic spin and pseudospin solutions to the Dirac equation for the Quadratic exponential-type potential plus Eckart potential and Yukawa-like tensor interaction**.** We have applied the approximation on the spin–orbit coupling term and the Yukawa potential. We used this scheme to obtain approximate analytical expressions for energies and eigenfunctions of the Yukawa potential for arbitrary spin–orbit quantum number k in the presence of spin symmetry, which is different from previous works.

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